# Helicoidal tracks

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## Abstract

Many formulae and utilities related to the description of a helix are presented. The L3 convention is assumed.

# 1   Introduction

The primary aim of these preliminary notes was to keep a confident traceback of all formulae used for the L3 SMD alignment. After few iterations they became a private consulting guide with some optimized tricks. At present, I hope that it will be useful for a wider audience.

You will find several figures across the text, but it consists mainly of algebraic games, based on few robust formulae. I realized that it is the right way, due to the presence of ambiguities in the sign of variables and in angle definitions. A geometrical view of the problem does not give in general the optimal answer and it is useful only "a posteriori".

Finally, these notes are not intended to be the "solution of everything" (some of the utilities are already present in the L3 software code), but a help for stand alone users like me, who spend hours and hours solving an infinite number of times the same problem, always getting different answers and never finding the best one.

# 2   Parametrization

A helicoidal track shows up whenever a charged particle is affected by a constant magnetic field. We will assume that this magnetic field goes along the positive Z axis of our reference system. The trajectory will be a circumference in the XY plane and the Z displacement will be proportional to the length of arc that is described in XY. We will talk about a "straight line in the SZ plane", where S means the variable associated to the previous arc length.

Given a vector $(p_x, p_y, p_z)$ we will adopt the following decomposition in spherical coordinates:

$$
\begin{aligned}
p_x &= p \cos \phi \, \sin \theta & (1)\\
p_y &= p \sin \phi \, \sin \theta & (2)\\
p_z &= p \cos \theta & (3)
\end{aligned}
$$

where $\phi$ is the azimuthal angle and $\theta$ the polar one. We will always consider that angles are given in the ranges:

$$
\begin{aligned}
\phi &\in [-\pi, \pi] & (4)\\
\theta &\in [0, \pi] & (5)
\end{aligned}
$$

## 2.1   XY plane

In the XY plane the movement is defined by a reference point, $(x_r, y_r)$, and 3 parameters $(C, \phi_0, \delta)$:

- $C$: the curvature of the track. The curvature is positive if the particle has a positive charge and negative if the charge is negative. Since the magnetic field points to Z>0, positive (negative) curvature means (anti)clock-wise rotation in the XY plane.

- $\phi_0$: the azimuthal angle of the momentum at the position of closest approach to the reference point.
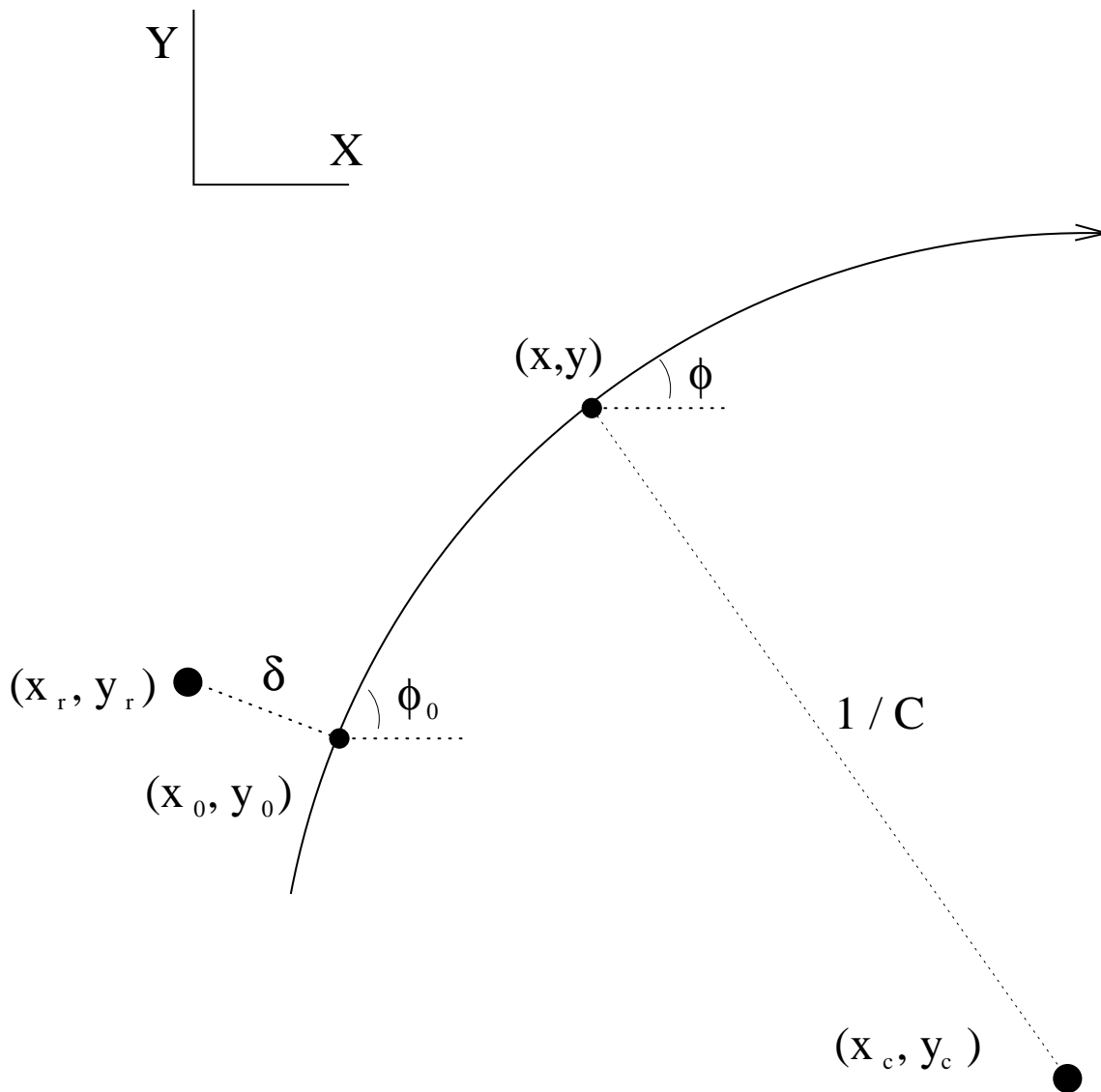
Figure 1: Following our convention, the projection of a helix in the XY plane is a circumference. The relevant parameters are shown in the figure.
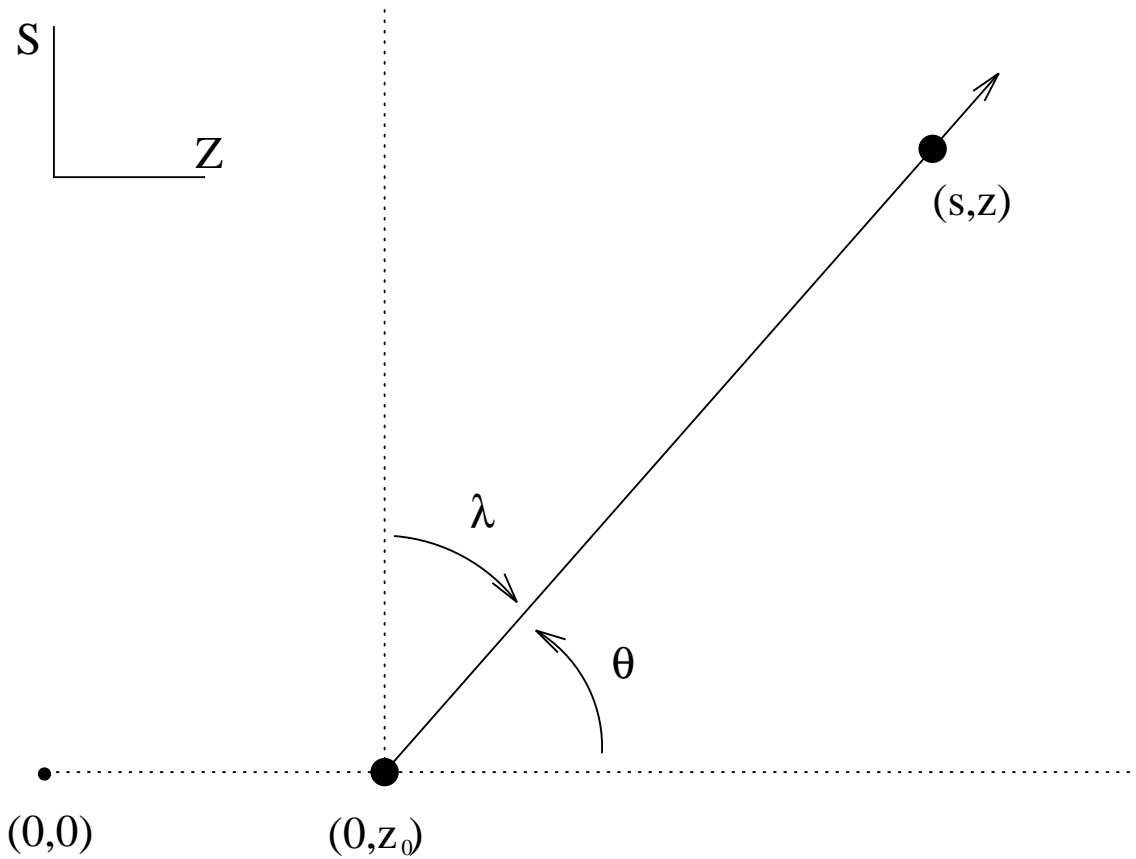
Figure 2: Following our convention, the projection of a helix in the SZ plane is a straight line: $z = z_0 + s \tan \lambda$. The relevant parameters are shown in the figure. The variable $s$ at a point $(x, y, z)$ is the arc length in the XY plane from $(x_0, y_0)$ to $(x, y)$. This also implies that $s = 0$ when $z = z_0$.

- $\delta$: the distance of closest approach to the reference point. This parameter is signed, with a convention such that the coordinates of closest approach $(x_0, y_0)$ are given by:

$$x_0 - x_r = -\delta \sin \phi_0 \qquad (6)$$
$$y_0 - y_r = +\delta \cos \phi_0 \qquad (7)$$
$$\Downarrow$$
$$\delta = -(x_0 - x_r) \sin \phi_0 + (y_0 - y_r) \cos \phi_0 \qquad (8)$$

If we define $\phi$ in a general way, as the azimuthal angle of the momentum vector at a point $(x, y, z)$, the following relation is satisfied:

$$x_c = x + R \sin \phi \qquad (9)$$
$$y_c = y - R \cos \phi \qquad (10)$$

where $R = 1/C$ and $(x_c, y_c)$ are the signed radius and the center of the circumference, respectively. In general, any two points $(x, y)$, $(x', y')$ on the track satisfy:

$$x + R \sin \phi = x' + R \sin \phi' \qquad (11)$$
$$y - R \cos \phi = y' - R \cos \phi' \qquad (12)$$

The circumference can be mathematically described in a single equation as follows:

$$(x - x_c)^2 + (y - y_c)^2 = R^2 \qquad (13)$$

where the center can be determined from the parameters at $(x_0, y_0)$ and formulae (9-10):

$$x_c = x_r + (R - \delta) \sin \phi_0 \qquad (14)$$
$$y_c = y_r - (R - \delta) \cos \phi_0 \qquad (15)$$

## 2.2 SZ plane

A crucial point in the L3 convention is that no use is made of $z_r$, the Z component of the reference point [1]. Every Z coordinate is referred to z=0. The straight line in the SZ plane is described by 2 parameters $(\tan \lambda, z_0)$:

- $\tan \lambda$: the slope in the SZ plane, $dz/ds$. It is a constant for a given track and it is directly related with the polar angle of the momentum vector, $\theta$:

$$\tan \lambda = \frac{p_z}{\sqrt{p_x^2 + p_y^2}} \qquad (16)$$

$$\tan \theta = \frac{1}{\tan \lambda} \qquad (17)$$

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[1] However, $z_r$ is included in the fit with a sizeable weight (20 $mm$ error, actually) due to the low quality of L3 Z measurements. The same is true in the $XY$ plane, but with a negligible effect (we expect at least $100\mu m$ precision at the vertex, which is much less than 20 $mm$).

- $z_0$: the Z position when the particle is at the distance of closest approach in the XY plane.

The equation of the trajectory in this plane is very simple:

$$z = z_0 + s \tan \lambda \tag{18}$$

where $s$ is the arc length when the particle travels from $(x_0, y_0)$ to $(x, y)$. The arc length is usually positive, but it can be negative if $(x, y)$ is "before" $(x_0, y_0)$ as the particle travels in time.

## 2.3 Practical answers to usual problems

- **How to find the $\phi$ angle at a given $(x, y)$ point?**

  The following relations are satisfied at $(x, y)$:

  $$x + R \sin \phi = x_0 + R \sin \phi_0 \tag{19}$$
  $$y - R \cos \phi = y_0 - R \cos \phi_0 \tag{20}$$

  Then, $\phi$ is given by:

  $$\phi = \textbf{\textit{atan2}} \left( \sin \phi_0 - C\ (x - x_0), \cos \phi_0 + C\ (y - y_0) \right) \tag{21}$$

  where $\textbf{\textit{atan2}}(y, x)$ is the FORTRAN function that finds the right angle in the XY plane for a given vector $(x, y)$. If we do not want to use $C$, let us define:

  $$\phi_s = \textbf{\textit{atan2}}\ (y - y_0, x - x_0) \tag{22}$$

  Substituting this new definition in (19-20) gives:

  $$\cos(\phi - \phi_s) = \cos(\phi_0 - \phi_s) \tag{23}$$
  $$\Downarrow$$
  $$\phi = \textbf{proxim}(2\ \phi_s - \phi_0, 0.) \tag{24}$$

  where $\textbf{proxim}(\phi, \xi)$ is a function that brings the angle $\phi$ into the $(-\pi + \xi, \pi + \xi)$ range. Its definition is exactly the following [1] :

  $$\textbf{proxim}(\phi, \xi) = \phi + 2\pi\ \textbf{\textit{nint}}(\frac{\xi - \phi}{2\pi}) \tag{25}$$

  where $\textbf{\textit{nint}}(x)$ is the closest integer to the real number $x$.

  Both formulae (21 and 24) give a good answer in the $C \to 0$ limit.

- **How to find the value of the curvature $C$ from $(x_0, y_0)$, $(x, y)$ and $\phi_0$ ?**

  Eliminating $\phi$ from expressions (19-20) we arrive to:

  $$C = \frac{2(x - x_0) \sin \phi_0 - 2(y - y_0) \cos \phi_0}{(x - x_0)^2 + (y - y_0)^2} \tag{26}$$

- **How to find the arc length from $(x_0, y_0)$ to $(x, y)$ ?**

  The exact answer is given by:

  $$s = -\frac{\Delta\phi}{C} \tag{27}$$

  where $\Delta\phi$ is $\phi - \phi_0$, the arc described in the $XY$ plane when we go from $(x_0, y_0, z_0)$ to $(x, y, z)$. There is a $2\pi$ ambiguity in $\Delta\phi$ as determined from $XY$ information only. This fact is always unrelevant, except for $s$. To avoid this kind of ambiguities we will restrict ourselves to solutions such that:

  $$\Delta\phi \in (-\pi, \pi) \tag{28}$$

  In this range equation (27) reads:

  $$s = \frac{\mathbf{proxim}(\phi_0 - \phi, 0.)}{C} \tag{29}$$

  However, this expression is not well behaved in the $C \to 0$ limit. The values of $\Delta\phi$ and $C$ can be really very small and the ratio will run into a lack of accuracy. There are other alternatives. From (19-20) we get:

  $$\alpha \equiv \sin(\Delta\phi) = -C(x - x_0) \cos \phi_0 - C(y - y_0) \sin \phi_0 \tag{30}$$
  $$\beta \equiv \cos(\Delta\phi) = 1 - C(x - x_0) \sin \phi_0 + C(y - y_0) \cos \phi_0 \tag{31}$$
  $$\Downarrow$$
  $$s = \frac{\mathbf{atan2}(-\alpha, \beta)}{C} \tag{32}$$

  which only makes use of the curvature. Finally, we can get a very interesting expression substituting $C = -\Delta\phi/s$ in (30):

  $$s = \frac{(x - x_0) \cos \phi_0 + (y - y_0) \sin \phi_0}{\mathbf{sinc}(\Delta\phi)} \tag{33}$$

  with $\mathbf{sinc}(x) = (\sin x)/x$. This formula is also well suited for the $C \to 0$ limit:

  $$s = (x - x_0) \cos \phi_0 + (y - y_0) \sin \phi_0 \tag{34}$$

7

- **How to find the $(x, y)$ point if we know the arc legnth ?**

  To avoid accuracy problems we should use the curvature $C$ in a clever way. Using formulae (30-31) and $\Delta\phi = -Cs$ we get:

$$x = x_0 + s \, \mathbf{sinc}\left(\frac{Cs}{2}\right) \cos\left(\phi_0 - \frac{Cs}{2}\right) \tag{35}$$

$$y = y_0 + s \, \mathbf{sinc}\left(\frac{Cs}{2}\right) \sin\left(\phi_0 - \frac{Cs}{2}\right) \tag{36}$$

# 3   Comments on L3 fitting

Given the fact that XY and Z measurements are almost decoupled [2], one can split the helix problem into two different ones: a fit to a circumference in XY plus a fit to a straight line in SZ.

The circumference fit is performed using the linear approach of reference [2] [3]:

$$\delta'_k = \delta + (x_k - x_r)\sin\phi_0 - (y_k - y_r)\cos\phi_0 - \frac{C}{2(1 - C\delta)}\left((x_k - x_r)^2 + (y_k - y_r)^2\right) \tag{37}$$

where $(x_k, y_k)$ are the coordinates of the *kth* point and $\delta'_k$ is its distance to the circumference. The $\chi^2$ to be minimized is:

$$\chi^2 = \sum_k w_k \, \delta'^2_k \tag{38}$$

where $w_k$ is the weight of the *kth* point. There exists an exact solution for the minimum of the $\chi^2$ in the variables $\delta$, $\phi$, $C/(1 - C\delta)$ [4]

The straight line fit in the SZ plane is performed considering uncertainties only in Z coordinates:

$$\chi^2 = \sum_j w_j \, (z_j - z_0 - s_j \tan\lambda)^2 \tag{39}$$

where $z_j$ is the Z coordinate of the *jth* point and $s_j$ is its arc length derived from the previous XY circumference fit. Neglecting the error on $s$ is reasonable if $\Delta_s \ll \Delta_z$, which seems to be the case in absence of microvertex information.

# 4   Change of reference point

How do track parameters change when the reference point moves from its initial value $(x_r, y_r, z_r)$ to a new one $(x'_r, y'_r, z'_r)$? This is a question to be answered here. We will also determine how does the

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[2] There is a little correlation in the case of the SMD due to the $2°$ tilt of the outer layer, but it may be neglected to simplify the problem.

[3] We will derive this formula in section 4.

[4] This is only true if the weights $w_k$ do not depend on $\delta$, $\phi$ or $C/(1-C\delta)$. This fact has not been taken into account in [2]. For instance, in the case of L3 the uncertainty goes along the azimuthal direction, and neglecting the dependence of the weights on the fitted parameters introduces an error of order $Cs$, and not $C\delta$. That gives a little bias for low momentum tracks.

covariance matrix change and some useful approximations to the problem. We do not do that for fun. There are obvious applications like, for instance, when the crossing point between the track and any given object has been found. Defining this crossing point as the new reference point gives access to all track parameters and errors there through the expressions of this section [5]. In this way we only need to make this delicate game once.

## 4.1  New parameters

The new parameters, $\delta'$ and $\phi_0'$, should satisfy:

$$x_r' + (R - \delta') \sin \phi_0' = x_r + (R - \delta) \sin \phi_0 \tag{40}$$
$$y_r' - (R - \delta') \cos \phi_0' = y_r - (R - \delta) \cos \phi_0 \tag{41}$$

or, in a more convenient form:

$$\frac{R - \delta'}{R - \delta} \sin \phi_0' = \sin \phi_0 - \frac{\Delta_x}{R - \delta} \tag{42}$$
$$\frac{R - \delta'}{R - \delta} \cos \phi_0' = \cos \phi_0 + \frac{\Delta_y}{R - \delta} \tag{43}$$

where $\Delta_x = x_r' - x_r$ and $\Delta_y = y_r' - y_r$

Due to the fact that in all practical cases $\text{sign}(R - \delta) = \text{sign}(R - \delta')$, $\phi_0'$ can be determined as follows, whithout any ambiguity:

$$\phi_0' = \mathbf{atan2} \left( \sin \phi_0 - \frac{\Delta_x}{R - \delta}, \cos \phi_0 + \frac{\Delta_y}{R - \delta} \right) \tag{44}$$

The value for $\delta'$ must be extracted by eliminating $\phi_0'$ from expressions (42), (43):

$$(R - \delta')^2 = (R - \delta)^2 - 2\Delta_x (R - \delta) \sin \phi_0 + 2\Delta_y (R - \delta) \cos \phi_0 + \Delta_x^2 + \Delta_y^2 \tag{45}$$

and applying the same assumption as before ($\text{sign}(R - \delta) = \text{sign}(R - \delta')$):

$$R - \delta' = (R - \delta) \sqrt{1 + \frac{-2\Delta_x \sin \phi_0 + 2\Delta_y \cos \phi_0}{R - \delta} + \frac{\Delta_x^2 + \Delta_y^2}{(R - \delta)^2}} \tag{46}$$

$$\Downarrow$$

$$\delta' = R - (R - \delta) \sqrt{1 + \frac{-2\Delta_x \sin \phi_0 + 2\Delta_y \cos \phi_0}{R - \delta} + \frac{\Delta_x^2 + \Delta_y^2}{(R - \delta)^2}} \tag{47}$$

Provided that $\phi_0'$ is already known there is a more convenient expression with a good behaviour in the $C \to 0$ limit:

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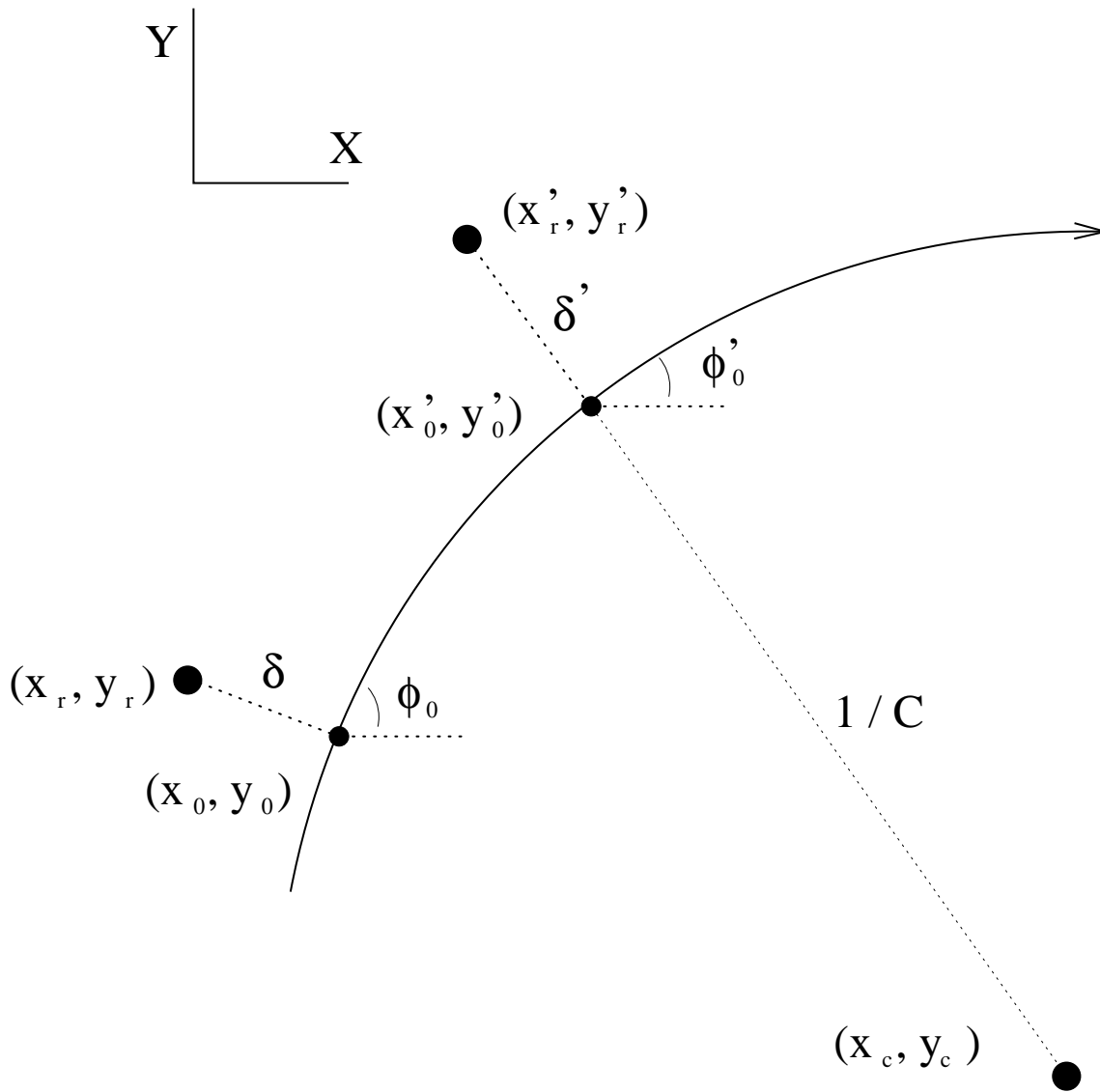[5] You simply have to determine $\phi_0'$ and impose $\delta' = 0$ in any formula of the section.

Figure 3: Change of parameters in XY when the reference point moves from $(x_r, y_r)$ to $(x'_r, y'_r)$.

$$\delta' = \delta + \Delta_x \sin \phi_0 - \Delta_y \cos \phi_0 + (\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \tan(\frac{\phi_0' - \phi_0}{2}) \tag{48}$$

The point of minimum approach to $(x_r', y_r')$ is:

$$x_0' = x_r' - \delta' \sin \phi_0' \tag{49}$$
$$y_0' = y_r' + \delta' \cos \phi_0' \tag{50}$$

and the corresponding Z coordinate is:

$$z_0' = z_0 + s \tan \lambda \tag{51}$$

where $s$ can be determined from the expressions in section 2.3.

## 4.2   Approximate solutions for $\delta'$

Formula (45) can be very well approximated in the limit in which $\delta' \ll R$ (that is, when the new reference point is very close to the track):

$$\delta' = \delta - \frac{C\delta^2}{2} + (\Delta_x \sin \phi_0 - \Delta_y \cos \phi_0) \ (1 - C\delta) - C \frac{\Delta_x^2 + \Delta_y^2}{2} \tag{52}$$

where the neglected term on the right hand side is $C\delta'^2/2$. This expression is even more useful when one realizes that equation (47) leads to computer problems in the $C \to 0$ limit whereas this one does not. Moreover, if we divide both sides of the previous equation by $(1 - C\delta)$ and we neglect $C\delta^2$ and $C\delta\delta'$ terms:

$$\delta' = \delta + \Delta_x \sin \phi_0 - \Delta_y \cos \phi_0 - \frac{C}{2(1 - C\delta)} \ \left(\Delta_x^2 + \Delta_y^2\right) \tag{53}$$

which is a very useful and simple expression. This approximation is the basis of the track reconstruction method (see section 3) in L3.

## 4.3   New $(C, \phi, \delta)$ covariance matrix

The new covariance matrix is built as follows (in a linear approximation):

$$V' = J \, V \, J^T \tag{54}$$

where $J$ is the jacobian matrix, $J^T$ is its transpose, $V$ is the original covariance matrix and $V'$ is the new one. The most economic way to find the relevant derivatives of the transformation is to differentiate expressions (40), (41):

$$-\frac{\partial \delta'}{\partial \phi_0} \ \sin \phi_0' + \frac{\partial \phi_0'}{\partial \phi_0} \ \cos \phi_0' \ (R - \delta') \ = \ +(R - \delta) \ \cos \phi_0 \tag{55}$$

$$-\frac{\partial \delta'}{\partial \phi_0} \ \cos \phi_0' - \frac{\partial \phi_0'}{\partial \phi_0} \ \sin \phi_0' \ (R - \delta') \ = \ -(R - \delta) \ \sin \phi_0 \tag{56}$$

$$-\frac{\partial \delta'}{\partial \delta} \ \sin \phi_0' + \frac{\partial \phi_0'}{\partial \delta} \ \cos \phi_0' \ (R - \delta') \ = \ -\sin \phi_0 \tag{57}$$

$$-\frac{\partial \delta'}{\partial \delta} \ \cos \phi_0' - \frac{\partial \phi_0'}{\partial \delta} \ \sin \phi_0' \ (R - \delta') \ = \ -\cos \phi_0 \tag{58}$$

$$-R^2 \ \sin \phi_0' - \frac{\partial \delta'}{\partial C} \ \sin \phi_0' + \frac{\partial \phi_0'}{\partial C} \ \cos \phi_0' \ (R - \delta') \ = \ -R^2 \ \sin \phi_0 \tag{59}$$

$$-R^2 \ \cos \phi_0' - \frac{\partial \delta'}{\partial C} \ \cos \phi_0' - \frac{\partial \phi_0'}{\partial C} \ \sin \phi_0' \ (R - \delta') \ = \ -R^2 \ \cos \phi_0 \tag{60}$$

Now it is easy to isolate all derivatives:

$$\frac{\partial C'}{\partial C} \ = \ 1 \tag{61}$$

$$\frac{\partial C'}{\partial \phi_0} \ = \ 0 \tag{62}$$

$$\frac{\partial C'}{\partial \delta} \ = \ 0 \tag{63}$$

$$\frac{\partial \phi_0'}{\partial C} \ = \ \frac{R^2 \ \sin(\phi_o' - \phi_0)}{R - \delta'} \tag{64}$$

$$\frac{\partial \phi_0'}{\partial \phi_0} \ = \ \frac{(R - \delta) \ \cos(\phi_o' - \phi_0)}{R - \delta'} \tag{65}$$

$$\frac{\partial \phi_0'}{\partial \delta} \ = \ \frac{\sin(\phi_o' - \phi_0)}{R - \delta'} \tag{66}$$

$$\frac{\partial \delta'}{\partial C} \ = \ R^2 \ (\cos(\phi_0' - \phi_0) - 1) \tag{67}$$

$$\frac{\partial \delta'}{\partial \phi_0} \ = \ -(R - \delta) \ \sin(\phi_o' - \phi_0) \tag{68}$$

$$\frac{\partial \delta'}{\partial \delta} \ = \ \cos(\phi_0' - \phi_0) \tag{69}$$

which is a very useful compressed form. However, for computational needs is better to work without high values of $R$ (that is, we want well behaved expressions in the straight line limit). Making use of $\Delta_x$, $\Delta_y$ to replace $\cos(\phi_o' - \phi_0)$ and $\sin(\phi_o' - \phi_0)$:

$$\sin(\phi_o' - \phi_0) \ = \ \frac{-C}{1 - C\delta'} \ (\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \tag{70}$$

$$\cos(\phi_o' - \phi_0) \ = \ \frac{1 - C\delta}{1 - C\delta'} + \frac{C}{1 - C\delta'} \ (-\Delta_x \sin \phi_0 + \Delta_y \cos \phi_0)$$

$$= \ 1 + \frac{C^2}{2(1 - C\delta')} \ (\delta'^2 - (\Delta_x + \delta \sin \phi_0)^2 - (\Delta_y - \delta \cos \phi_0)^2) \tag{71}$$

$$\Downarrow \ \Downarrow$$

12

$$\frac{\partial C'}{\partial C} = 1 \tag{72}$$

$$\frac{\partial C'}{\partial \phi_0} = 0 \tag{73}$$

$$\frac{\partial C'}{\partial \delta} = 0 \tag{74}$$

$$\frac{\partial \phi_0'}{\partial C} = -\frac{1}{(1 - C\delta')^2} (\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \tag{75}$$

$$\frac{\partial \phi_0'}{\partial \phi_0} = \left(\frac{1 - C\delta}{1 - C\delta'}\right) + \frac{C^2(1 - C\delta)}{2(1 - C\delta')^2} (\delta'^2 - (\Delta_x + \delta \sin \phi_0)^2 - (\Delta_y - \delta \cos \phi_0)^2) \tag{76}$$

$$\frac{\partial \phi_0'}{\partial \delta} = -\frac{C^2}{(1 - C\delta')^2} (\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \tag{77}$$

$$\frac{\partial \delta'}{\partial C} = \frac{1}{2(1 - C\delta')} (\delta'^2 - (\Delta_x + \delta \sin \phi_0)^2 - (\Delta_y - \delta \cos \phi_0)^2) \tag{78}$$

$$\frac{\partial \delta'}{\partial \phi_0} = \frac{1 - C\delta}{1 - C\delta'} (\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \tag{79}$$

$$\frac{\partial \delta'}{\partial \delta} = 1 + \frac{C^2}{2(1 - C\delta')} (\delta'^2 - (\Delta_x + \delta \sin \phi_0)^2 - (\Delta_y - \delta \cos \phi_0)^2) \tag{80}$$

and in the straight track limit:

$$\frac{\partial C'}{\partial C} = 1 \tag{81}$$

$$\frac{\partial C'}{\partial \phi_0} = 0 \tag{82}$$

$$\frac{\partial C'}{\partial \delta} = 0 \tag{83}$$

$$\frac{\partial \phi_0'}{\partial C} = -(\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \tag{84}$$

$$\frac{\partial \phi_0'}{\partial \phi_0} = 1 \tag{85}$$

$$\frac{\partial \phi_0'}{\partial \delta} = 0 \tag{86}$$

$$\frac{\partial \delta'}{\partial C} = \frac{1}{2} (\delta'^2 - (\Delta_x + \delta \sin \phi_0)^2 - (\Delta_y - \delta \cos \phi_0)^2) \tag{87}$$

$$\frac{\partial \delta'}{\partial \phi_0} = (\Delta_x \cos \phi_0 + \Delta_y \sin \phi_0) \tag{88}$$

$$\frac{\partial \delta'}{\partial \delta} = 1 \tag{89}$$

## 4.4   New SZ covariance matrix

All derivatives are in this case much simpler:

$$\frac{\partial \tan \lambda'}{\partial \tan \lambda} = 1 \qquad (90)$$

$$\frac{\partial \tan \lambda'}{\partial z_0} = 0 \qquad (91)$$

$$\frac{\partial z_0'}{\partial \tan \lambda} = s \qquad (92)$$

$$\frac{\partial z_0'}{\partial z_0} = 1 \qquad (93)$$

which gives the covariance matrix:

$$V_{sz}' = \begin{pmatrix} V_{\tan \lambda \tan \lambda} & s\, V_{\tan \lambda \tan \lambda} + V_{\tan \lambda\; z0} \\ s\, V_{\tan \lambda \tan \lambda} + V_{\tan \lambda\; z0} & s^2\, V_{\tan \lambda \tan \lambda} + 2s\, V_{\tan \lambda\; z0} + V_{z0\; z0} \end{pmatrix} \qquad (94)$$

# 5 Track crossing a cylinder parallel to Z

The crossing point $(x_0', y_0', z_0')$ is defined by the conditions:

$$x_0' + R\, \sin \phi_0' = x_0 + R\, \sin \phi_0 \qquad (95)$$
$$y_0' - R\, \cos \phi_0' = y_0 - R\, \cos \phi_0 \qquad (96)$$
$$z_0' = z_0 + s\, \tan \lambda \qquad (97)$$
$$\rho^2 = (x_0' - x_\rho)^2 + (y_0' - y_\rho)^2 \qquad (98)$$

where $(x_\rho, y_\rho)$ is the center of the circumference to be crossed and $\rho$ is its radius. To solve the problem we will use the following definitions:

$$\phi_\rho = \mathbf{atan2}\,(\sin \phi_0 - C(x_\rho - x_0), \cos \phi_0 + C(y_\rho - y_0)) \qquad (99)$$
$$\phi_c = \mathbf{atan2}\,(y_0' - y_\rho, x_0' - x_\rho) \qquad (100)$$
$$\gamma = \frac{2(x_\rho - x_0)\sin \phi_0 - 2(y_\rho - y_0)\cos \phi_0 - C\rho^2 - C((x_\rho - x_0)^2 + (y_\rho - y_0)^2)}{2\rho\sqrt{(\sin \phi_0 - C(x_\rho - x_0))^2 + (\cos \phi_0 + C(y_\rho - y_0))^2}} \qquad (101)$$

The solution can be derived from:

$$\sin(\phi_c - \phi_\rho) = \gamma \qquad (102)$$
$$\qquad (103)$$

which has a meaning if $\mid \gamma \mid < 1$ and gives two possible angles $\phi_c$ (as expected). The values for $(x_0', y_0')$ are:

$$x_0' = x_\rho + \rho\, \cos \phi_c \qquad (104)$$
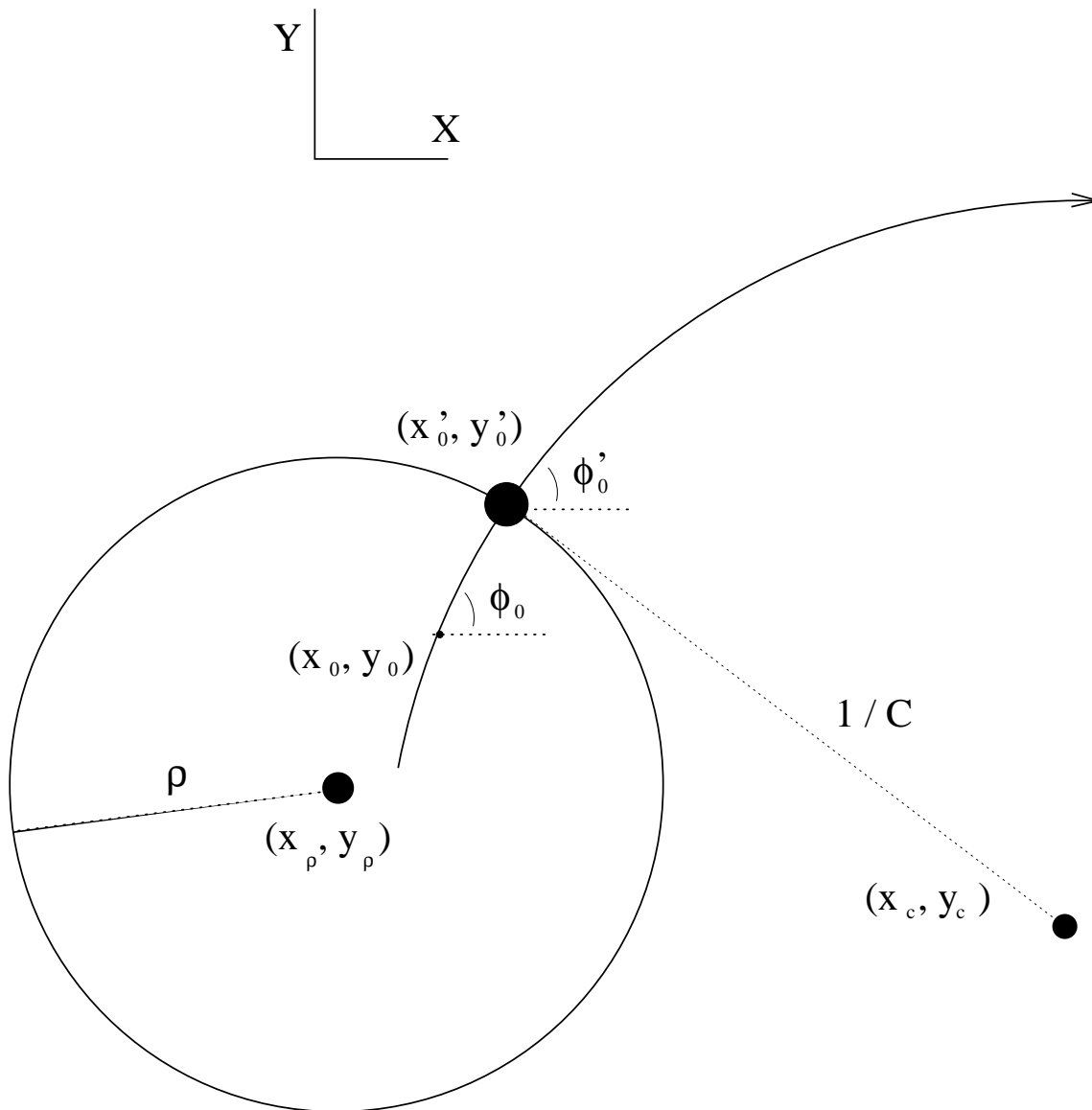$$y_0' = y_\rho + \rho\, \sin \phi_c \qquad (105)$$

14

Figure 4: Track crossing a circumference in the XY plane.

but usually we will be interested in a solution with positive arc length:

$$(x_0' - x_0) \cos \phi_0 + (y_0' - y_0) \sin \phi_0 > 0 \qquad (106)$$

A particular case is that of a cylinder with $\rho^2 < R^2$ and $(x_0 - x_\rho)^2 + (y_0 - y_\rho)^2 < \rho^2$, that is, a cylinder with its center very close to the distance of minimum approach. In this case there is only one solution with $s > 0$:

$$\phi_c = \phi_\rho + \arcsin(\gamma) \qquad (107)$$

# 6 Track crossing a plane

The crossing point $(x_0', y_0', z_0')$ is defined by the following conditions:

$$x_0' = x_0 + s \, \textbf{sinc} \left( \frac{Cs}{2} \right) \cos \left( \phi_0 - \frac{Cs}{2} \right) \qquad (108)$$

$$y_0' = y_0 + s \, \textbf{sinc} \left( \frac{Cs}{2} \right) \sin \left( \phi_0 - \frac{Cs}{2} \right) \qquad (109)$$

$$z_0' = z_0 + s \, \tan \lambda \qquad (110)$$

$$0 = (x_p - x_0') \, v_x + (y_p - y_0') \, v_y + (z_p - z_0') \, v_z \qquad (111)$$

where $(x_p, y_p, z_p)$ is a given point on the plane and $(v_x, v_y, v_z)$ is an **unitary** vector perpendicular to it. To solve the problem we will define:

$$\phi_v = \textbf{atan2} \, (v_y, v_x) \qquad (112)$$

$$v_t = \sqrt{v_x^2 + v_y^2} \qquad (113)$$

$$d_p = (x_p - x_0) \, v_x + (y_p - y_0) \, v_y + (z_p - z_0) \, v_z \qquad (114)$$

where $d_p$ is in fact the signed distance from $(x_0, y_0, z_0)$ to the plane. Then we can get an equation in the variable $s$:

$$s = \frac{d_p}{v_t \, \textbf{sinc} \left( \frac{Cs}{2} \right) \cos \left( \frac{Cs}{2} + \phi_v - \phi_0 \right) + v_z \, \tan \lambda} \qquad (115)$$

The equation can be solved iteratively assuming the parameter $Cs$ to be small:

$$s_0 = \frac{d_p}{v_t \, \cos(\phi_v - \phi_0) + v_z \, \tan \lambda} \qquad (116)$$

$$s_1 = \frac{d_p}{v_t \, \textbf{sinc} \left( \frac{Cs_0}{2} \right) \cos \left( \frac{Cs_0}{2} + \phi_v - \phi_0 \right) + v_z \, \tan \lambda} \qquad (117)$$

$$\vdots$$

$$s_i = \frac{d_p}{v_t \, \textbf{sinc} \left( \frac{Cs_{i-1}}{2} \right) \cos \left( \frac{Cs_{i-1}}{2} + \phi_v - \phi_0 \right) + v_z \, \tan \lambda} \qquad (118)$$
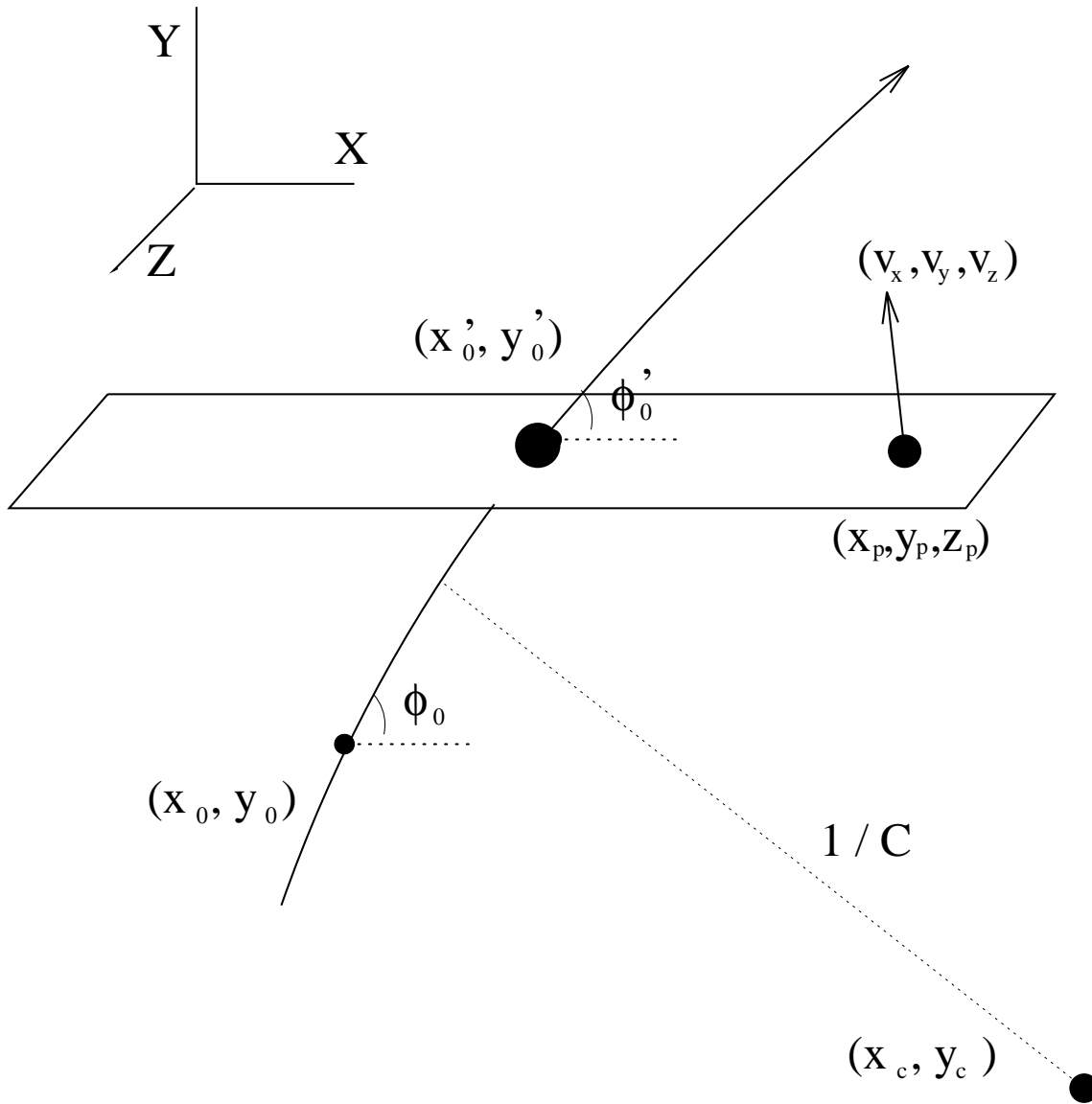
16

Figure 5: Track crossing a plane.

where one should check that $\mid Cs \mid < \pi$ at every step. In the special case in which $v_z = 0$ the equation can be exactly solved in the variable $Cs$:

$$\sin(Cs + \phi_v - \phi_0) = \sin(\phi_v - \phi_0) + C\,d_p \tag{119}$$

which gives two possible solutions (but we choose the one with minimum arc length). The final answer for $s$ comes from substitution of $Cs$ in expression (115) [6].

# 7   Including new points in a track

Let us assume that the track goes through the point $(x_{fix}, y_{fix}, z_{fix})$ with some uncertainty. Let us also assume that all we know from the track are its parameters and the covariance matrix. Even if so, it is is possible to include that point in the fit assuming a linearized behaviour around the minimum. This hypothesis is almost true if the variation of the track parameters in the whole game is very small. If this is not true, even the transport of the covariance matrix has no meaning.

## 7.1   XY plane

The $\chi^2$ in absence of new points is:

$$\chi^2 = x_i\,S_{ij}\,x_j \tag{120}$$
$$x_1 = C - C^0 \tag{121}$$
$$x_2 = \phi_0 - \phi_0^0 \tag{122}$$
$$x_3 = \delta - \delta^0 \tag{123}$$

where a sum on repeated indices is assumed. $S_{ij}$ is the inverse of the covariance matrix, $V_{ij}$, and $C^0$, $\phi_0^0$ and $\delta^0$ are the actual values of the track parameters (the obvious minimum of the $\chi^2$ is at $C = C^0$, $\phi_0 = \phi_0^0$, $\delta = \delta^0$).

Let us define a second set of parameters. For that, we will use $(x_{fix}, y_{fix}, z_{fix})$ as reference point, the one which is going to be included in the fit. From section 4 we know how to extract the new azimuthal angle and the new distance of closest approach from the initial set of parameters:

$$\Delta_x = x_{fix} - x_r \tag{124}$$
$$\Delta_y = y_{fix} - y_r \tag{125}$$
$$\downarrow$$
$$\phi_0' = \textbf{atan2}\left(\sin\phi_0 - \frac{\Delta_x}{R - \delta}, \cos\phi_0 + \frac{\Delta_y}{R - \delta}\right) \tag{126}$$
$$\delta' = \delta - \frac{C\delta^2}{2} + (\Delta_x \sin\phi_0 - \Delta_y \cos\phi_0)\,(1 - C\delta) - C\,\frac{\Delta_x^2 + \Delta_y^2}{2} \tag{127}$$

where we have neglected the term $C\,\delta'^2/2$, given the nature of the problem. We can also establish the $\chi^2$ in this second reference system:

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[6] To gain acccuracy do not divide $Cs$ by C.

$$\chi^2 = x_i' S_{ij}' x_j' \tag{128}$$

$$x_1' = C' - C^{0\prime} \equiv C - C^0 \tag{129}$$

$$x_2' = \phi_0' - \phi_0^{0\prime} \tag{130}$$

$$x_3' = \delta' - \delta^{0\prime} \tag{131}$$

where $S_{ij}'$ is the inverse of $V_{ij}'$, the covariance matrix after transport. In the linear approximation used:

$$x_i' = \left.\frac{\partial x_i'}{\partial x_j}\right|_{x=0} x_j \tag{132}$$

$$V_{ij}' = \left.\frac{\partial x_i'}{\partial x_k}\right|_{x=0} \left.\frac{\partial x_j'}{\partial x_l}\right|_{x=0} V_{kl} \tag{133}$$

where the derivatives can be found from equations (72) to (80) in section 4 by replacing:

$$\partial C' \rightarrow \partial x_1' \tag{134}$$

$$\partial \phi_0' \rightarrow \partial x_2' \tag{135}$$

$$\partial \delta' \rightarrow \partial x_3' \tag{136}$$

$$\partial C \rightarrow \partial x_1 \tag{137}$$

$$\partial \phi_0 \rightarrow \partial x_2 \tag{138}$$

$$\partial \delta \rightarrow \partial x_3 \tag{139}$$

The new point should belong to the track. That means that $\delta'$ is zero within its uncertainty, $\sigma$. In the second reference system, the $\chi^2$ changes as follows:

$$\chi^2 = x_i' S_{ij}' x_j' + \left(\frac{\delta'}{\sigma}\right)^2 \tag{140}$$

$$= x_i' S_{ij}' x_j' + \left(\frac{\delta^{0\prime} + x_3'}{\sigma}\right)^2 \tag{141}$$

which leads to the solution:

$$V_{kj}'^{\,new} = \left(S_{kj}' + \frac{1}{\sigma^2} g_{k3} g_{j3}\right)^{-1} \tag{142}$$

$$\Downarrow$$

$$x_k' = -V_{k3}'^{\,new} \frac{\delta^{0\prime}}{\sigma^2} \tag{143}$$

where $g_{ij}$ is the identity matrix (1 if $i = j$, 0 otherwise).

In the general case there will be several points in the fit. We can not make things so easy because the $\chi^2$ must be established in only one system. Then, it is better to establish it in the original frame:

$$\chi^2 = x_i \, S_{ij} \, x_j + \sum_{k=1}^{N} \left( \frac{\delta_k'}{\sigma_k} \right)^2 \tag{144}$$

$$= x_i \, S_{ij} \, x_j + \sum_{k=1}^{N} \left( \frac{\delta_k^{0\prime} + \frac{\partial x_{3k}'}{\partial x_j} \Big|_{x=0} x_j}{\sigma_k} \right)^2 \tag{145}$$

for N points $(\phi_{0k}', \delta_k'), k = 1, N$. Finding the minimum of this $\chi^2$ leads to:

$$V_{ij}^{new} = \left( S_{ij} + \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \frac{\partial x_{3k}'}{\partial x_i} \Big|_{x=0} \frac{\partial x_{3k}'}{\partial x_j} \Big|_{x=0} \right)^{-1} \tag{146}$$

$$\Downarrow$$

$$x_i = -V_{ij}^{new} \sum_{k=1}^{N} \frac{\delta_k^{0\prime}}{\sigma_k^2} \frac{\partial x_{3k}'}{\partial x_j} \Big|_{x=0} \tag{147}$$

## 7.2   SZ plane

The problem is similar to the XY case, but slightly simpler. We will do everything in the initial reference system:

$$\chi^2 = x_i \, T_{ij} \, x_j + \left( \frac{x_2 + s x_1 + z_0^0 + s \tan \lambda^0 - z_{fix}}{\sigma} \right)^2 \tag{148}$$

$$x_1 = \tan \lambda - \tan \lambda^0 \tag{149}$$

$$x_2 = z_0 - z_0^0 \tag{150}$$

where $s$ is the arc length described in the XY plane and $T_{ij}$ is the inverse of $W_{ij}$, the covariance matrix in the SZ plane. The solution is:

$$W_{ij}^{new} = \begin{pmatrix} T_{11} + \frac{s^2}{\sigma^2} & T_{12} + \frac{s}{\sigma^2} \\ T_{21} + \frac{s}{\sigma^2} & T_{22} + \frac{1}{\sigma^2} \end{pmatrix}^{-1} \tag{151}$$

$$\Downarrow$$

$$x_1 = -(W_{11}^{new} s + W_{12}^{new}) \frac{z_0^0 + s \tan \lambda^0 - z_{fix}}{\sigma^2} \tag{152}$$

$$x_2 = -(W_{21}^{new} s + W_{22}^{new}) \frac{z_0^0 + s \tan \lambda^0 - z_{fix}}{\sigma^2} \tag{153}$$

Finally, the most general solution for several points is:

$$W_{ij}^{new} = \begin{pmatrix} T_{11} + \sum_{k=1}^{N} \frac{s_k^2}{\sigma_k^2} & T_{12} + \sum_{k=1}^{N} \frac{s_k}{\sigma_k^2} \\ T_{21} + \sum_{k=1}^{N} \frac{s_k}{\sigma_k^2} & T_{22} + \sum_{k=1}^{N} \frac{1}{\sigma_k^2} \end{pmatrix}^{-1} \tag{154}$$

$$\Downarrow$$

$$x_1 = -\sum_{k=1}^{N}(W_{11}^{new} s_k + W_{12}^{new})\frac{z_{0k}^0 + s_k \tan \lambda^0 - z_{kfix}}{\sigma_k^2} \tag{155}$$

$$x_2 = -\sum_{k=1}^{N}(W_{21}^{new} s_k + W_{22}^{new})\frac{z_{0k}^0 + s_k \tan \lambda^0 - z_{kfix}}{\sigma_k^2} \tag{156}$$

# 8 Constraining a track to a point

## 8.1 XY plane

A very particular case of the previous one happens when $\sigma \to 0$, that is, when we want to fix the point to be on the track. That implies that $\delta'$ is zero. The $\chi^2$ should be written in a different way:

$$\chi^2 = x_i S_{ij} x_j + \xi \delta' \tag{157}$$

where $\xi$ is a Lagrange multiplier needed to impose the $\delta' = 0$ constraint. The solution is:

$$x_i = -\frac{\delta_0' V_{ij} \left.\frac{\partial x_3'}{\partial x_j}\right|_{x=0}}{\left.\frac{\partial x_3'}{\partial x_k}\right|_{x=0} \left.\frac{\partial x_3'}{\partial x_l}\right|_{x=0} V_{kl}} \tag{158}$$

Another option is to go to the system in which the fixed point is the reference point. There, the $\chi^2$ is again conceptually simpler:

$$\chi^2 = x_i' S_{ij}' x_j' + \xi \delta' \tag{159}$$

$$\tag{160}$$

with the solution:

$$C = C^0 - \frac{V_{13}'}{V_{33}'}\delta^{0\prime} \tag{161}$$

$$\phi_0' = \phi_0^{0\prime} - \frac{V_{23}'}{V_{33}'}\delta^{0\prime} \tag{162}$$

$$\delta' = 0 \tag{163}$$

The same result would have been obtained if we simply solve equation (143) with $\delta' = 0$ and $\sigma = 0$. However, the covariance matrix in that limit is:

$$V_{ij}'^{new} = \begin{pmatrix} \frac{S_{22}'}{S_{11}'S_{22'}-S_{12}'S_{12}'} & \frac{-S_{12}'}{S_{11}'S_{22}'-S_{12}'S_{12}'} & 0 \\ \frac{-S_{12}'}{S_{11}'S_{22}'-S_{12}'S_{12}'} & \frac{S_{11}'}{S_{11}'S_{22'}-S_{12}'S_{12}'} & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{164}$$

which must be used carefully (it is a singular matrix).

## 8.2 SZ plane

The solution is:

$$\tan \lambda = \tan \lambda^0 - \frac{sW_{11} + W_{12}}{s^2 W_{11} + 2sW_{12} + W_{22}} (z_0^0 + s \tan \lambda^0 - z_{fix}) \tag{165}$$

$$z_0 = z_{fix} - s \tan \lambda \tag{166}$$

And the new covariance matrix (also singular) in the limit $\sigma \to 0$ is:

$$W_{ij}^{new} = \frac{1}{T_{11} - 2sT_{12} + s^2 T_{22}} \begin{pmatrix} 1 & -s \\ -s & s^2 \end{pmatrix} \tag{167}$$

# 9 Common vertex for several tracks in XY

To find the common point of several tracks (at least 2) in the XY plane we would like to minimize for all tracks approximation (52):

$$\delta_i' = \delta_i - \frac{C_i \delta_i^2}{2} + (\Delta_x \sin \phi_{0i} - \Delta_y \cos \phi_{0i}) \ (1 - C_i \delta_i) - C_i \frac{\Delta_x^2 + \Delta_y^2}{2} \tag{168}$$

where $\delta_i'$ is the distance of closest approach of the *ith* track to the desired reference point $(x_r', y_r')$. The idea is that $\Delta_x$ and $\Delta_y$ are the parameters to be determined, which are the difference between the old reference point and the new one. The neglected term on the right hand side is $C_i \delta_i'^2/2$ which is expected to be very small. The $\chi^2$ is:

$$\chi_{xy}^2 = \sum_{i=1}^{N} \left( \frac{\delta_i'}{\sigma_i} \right)^2 \tag{169}$$

where $\sigma_i$ is the error on $\delta_i'$. The best way is to linearize the problem, by choosing the first reference point $(x_r, y_r)$ close to the final one. If that is the case, $C_i (\Delta_x^2 + \Delta_y^2)$ terms may be neglected. In the first step we minimize the following $\chi^2$:

$$\alpha_i = \frac{2 \delta_i - C_i \delta_i^2}{2 \sigma_i} \tag{170}$$

$$\beta_i = \frac{\sin \phi_{0i} (1 - C_i \delta_i)}{\sigma_i} \tag{171}$$

$$\gamma_i = -\frac{\cos \phi_{0i} (1 - C_i \delta_i)}{\sigma_i} \tag{172}$$

$$\downarrow$$

$$\chi_{xy}^2 = \sum_{i=1}^{N} (\alpha_i + \beta_i \Delta_x + \gamma_i \Delta_y)^2 \tag{173}$$

with the solution:

22

$$V_{\Delta x \Delta y} \;=\; \left( \begin{array}{cc} \sum \beta_i^2 & \sum \beta_i \gamma_i \\ \sum \beta_i \gamma_i & \sum \gamma_i^2 \end{array} \right)^{-1} \tag{174}$$

$$\left( \begin{array}{c} \Delta_x \\ \Delta_y \end{array} \right) \;=\; V_{\Delta x \Delta y} \left( \begin{array}{c} -\sum \alpha_i \beta_i \\ -\sum \alpha_i \gamma_i \end{array} \right) \tag{175}$$

The error on each $\delta_i'$ induced by linearization is $C_i \frac{\Delta_x^2 + \Delta_y^2}{2}$. If we have not enough accuracy the process should be iterated redefining $\alpha_i$ until we get the required precision:

$$\alpha_i \;=\; \frac{2\delta_i - C_i \delta_i^2}{2\sigma_i} - C_i \frac{\Delta_{0x}^2 + \Delta_{oy}^2}{2\sigma_i} \tag{176}$$

where $\Delta_{0x}$ and $\Delta_{0y}$ come from the previous iteration.

# 10  Common vertex for several tracks in space

The $\chi^2$ contribution in the SZ plane is the following:

$$\chi_z^2 \;=\; \sum_{i=1}^{N} \left( \frac{\Delta_Z - z_{0i} - \frac{\tan \lambda_i}{\mathbf{sinc}(C_i s_i)} \left( \Delta_x \cos \phi_{0i} + \Delta_y \sin \phi_{0i} \right)}{\sigma_{zi}} \right)^2 \tag{177}$$

where $s_i$ is the arc length from a displacement given by $(\Delta_x, \Delta_y)$ and $\sigma_{zi}$ is the error on $\Delta_Z$ from the *ith* track. Now $\Delta_Z$ is not the difference with respect to the $z$ reference point, but with respect to $z = 0$. In the formula above $C\delta'$ terms have been neglected. We have to minimize the global $\chi^2$:

$$\chi^2 = \chi_{xy}^2 + \chi_{sz}^2 \tag{178}$$

Then the problem can be still linearized in an iterative process:

$$\alpha_i \;=\; \frac{2\delta_i - C_i \delta_i^2}{2\sigma_i} - C_i \frac{\Delta_{0x}^2 + \Delta_{oy}^2}{2\sigma_i} \tag{179}$$

$$\beta_i \;=\; \frac{\sin \phi_{0i} \, (1 - C_i \delta_i)}{\sigma_i} \tag{180}$$

$$\gamma_i \;=\; -\frac{\cos \phi_{0i} \, (1 - C_i \delta_i)}{\sigma_i} \tag{181}$$

$$\xi_i \;=\; \frac{\cos \phi_{0i} \tan \lambda_i}{\mathbf{sinc}(C_i s_{0i}) \, \sigma_{zi}} \tag{182}$$

$$\eta_i \;=\; \frac{\sin \phi_{0i} \tan \lambda_i}{\mathbf{sinc}(C_i s_{0i}) \, \sigma_{zi}} \tag{183}$$

$$\downarrow$$

$$V_{\Delta x \Delta y \Delta z} = \begin{pmatrix} \sum(\beta_i^2 + \xi_i^2) & \sum(\beta_i \gamma_i + \xi_i \eta_i) & -\sum \frac{\xi_i}{\sigma_{zi}} \\ \sum(\beta_i \gamma_i + \xi_i \eta_i) & \sum(\gamma_i^2 + \eta_i^2) & -\sum \frac{\eta_i}{\sigma_{zi}} \\ -\sum \frac{\xi_i}{\sigma_{zi}} & -\sum \frac{\eta_i}{\sigma_{zi}} & \sum \frac{1}{\sigma_{zi}^2} \end{pmatrix}^{-1} \tag{184}$$

$$\begin{pmatrix} \Delta_x \\ \Delta_y \\ \Delta_z \end{pmatrix} = V_{\Delta x \Delta y \Delta z} \begin{pmatrix} -\sum(\alpha_i \beta_i + z_{0i} \frac{\xi_i}{\sigma_{zi}}) \\ -\sum(\alpha_i \gamma_i + z_{0i} \frac{\eta_i}{\sigma_{zi}}) \\ \sum \frac{z_{0i}}{\sigma_{zi}^2} \end{pmatrix} \tag{185}$$

where $\Delta_{0x}$, $\Delta_{0y}$ and $s_{0i}$ are the results of the previous iteration and they are set to zero in the fist step.

# References

[1] CERN CN Division, Writeups of the CERN Program Library (1992)

[2] V. Karimäki, "Fast trajectory fitting in uniform field", UA1/TN/82-24 (1982);
    V. Karimäki, "Effective Circle Fitting for Particle Trajectories", HU-SEFT-1990-04 (1990)